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Time asymmetric quantum theory and the ambiguity of the Z-boson mass and width

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Abstract. Relativistic Gamow vectors emerge naturally in a time asymmetric quantum theory as the covariant kets associated to the resonance pole $s = s_R$ in the second sheet of the analytically continued S-matrix. They are eigenkets of the self-adjoint mass operator with complex eigenvalue $\sqrt{s_R}$ and have exponential time evolution with lifetime $\tau = -\hbar/2\mathrm{Im}\sqrt{s_R}$. If one requires that the resonance width Γ (defined by the Breit-Wigner lineshape) and the resonance lifetime τ always and exactly fulfill the relation $\Gamma = \hbar/\tau$, then one is lead to the following parameterization of s_R in terms of resonance mass M_R and width Γ_R: $s_R = (M_R - i\Gamma/2)^2$. Applying this result to the Z-boson implies that $M_R \approx M_Z - 26\mathrm{MeV}$ and $\Gamma_R \approx \Gamma_Z - 1.2\mathrm{MeV}$ are the mass and width of the Z-boson and not the particle data values (M_Z, Γ_Z) or any other parameterization of the Z-boson lineshape. Furthermore, the transformation properties of these Gamow kets show that they furnish an irreducible representation of the causal Poincaré semigroup, defined as a semi-direct product of the homogeneous Lorentz group with the *semigroup* of space-time translations into the forward light cone. Much like Wigner's unitary irreducible representations of the Poincaré group which describe stable particles, these irreducible semigroup representations can be characterized by the spin-mass values $(j, s_R = (M_R - i\Gamma/2)^2)$.

1 Introduction and motivation

The meaning of unstable elementary particles and/or resonances – in particular in the relativistic domain – has always been a subject of controversy and debates which flare-up whenever new phenomena compel us to re-examine our old ideas and prejudices. Recently it was the line shape of the Z-boson in the analyses of the LEP and SLC data of $e\bar{e} \to f\bar{f}(+n\gamma)$ that gave rise to the revision of old ideas. Two different approaches have been used in the determination of the line shape and the definition of the line shape parameters [1,2]. The first and popular approach, which practically all experimental analyses of the LEP and SLC data follow [3], is based on the on-shell definition of mass M_Z and width Γ_Z. Mass and width are defined in perturbation theory by the self-energy of the Z-boson propagator. The *on-shell definition* of mass and width defines the (real) mass M_Z as the renormalized mass in the on-shell renormalization scheme by the real part of the self-energy. This choice of M_Z as the mass of the Z is arbitrary. The s-dependent width $\Gamma_Z(\mathsf{s})$ (which is not a parameter of the standard model but a derived quantity) is given by the imaginary part of the self-energy in terms of the parameters of the standard model and M_Z, and thus suffers from the same degree of arbitrariness. In this on-shell approach, the (radiation corrected) cross sections around the Z peak are fitted to a Breit-Wigner amplitude with energy dependent width given by

$$a_j(\mathsf{s}) = -\frac{\sqrt{\mathsf{s}}\sqrt{\Gamma_e(\mathsf{s})\Gamma_f(\mathsf{s})}}{\mathsf{s} - M_Z^2 + i\sqrt{\mathsf{s}}\Gamma_Z(\mathsf{s})} \approx \frac{R_Z}{\mathsf{s} - M_Z^2 + i\frac{\mathsf{s}}{M_Z}\Gamma_Z}\,, \quad (1)$$

where for the Z boson propagator (neglecting the Fermion mass)

$$\sqrt{\mathsf{s}}\Gamma_Z(\mathsf{s}) = \frac{\mathsf{s}}{M_Z}\Gamma_Z \quad \text{and} \quad R_Z \equiv \sqrt{\Gamma_e\Gamma_f}\frac{\mathsf{s}}{M_Z} \quad (2)$$

have been used.

Once the arbitrariness of the on-shell renormalization scheme [4–7] and its problems with gauge invariance of M_Z and Γ_Z [8,9] were realized, a second approach to the Z-boson line shape was suggested. This was based on the S-matrix definition of the mass and width for an unstable particle with spin j by the pole position $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$ of the resonance pole on the second sheet of the j-th partial S-matrix element (or equivalently the position s_R of the propagator pole). With this definition, the j-th partial amplitude for the Z-boson is again given by a Breit-Wigner amplitude

$$a_j(\mathsf{s}) = \frac{R_Z}{\mathsf{s} - \left(M_R - i\frac{\Gamma_R}{2}\right)^2} = \frac{R_Z}{\mathsf{s} - \mathsf{s}_R}\,, \quad -\infty_{II} < \mathsf{s} < +\infty\,. \quad (3)$$

Since the S-matrix pole is in the second Riemann sheet the values of s should presumably also extend over the entire real axis in the second sheet. This makes a difference *not* for the physical (positive) values of s along the cut but only for the negative values as indicated by $-\infty_{II}$ in (3).

Usually the range of s is not stated and may often be presumed to extend over the values along the cut only, $\mathsf{s}_0 = (m_e + m_{\bar{e}})^2 < \mathsf{s} < \infty$, but it will turn out below that s should range as stated in (3). The width Γ and mass M_R are now the fixed basic S-matrix parameters, independent of the energy s or a particular renormalization scheme. According to the results of [5–7] the two definitions differ in value by an amount exceeding the experimental error:

$$M_R \approx M_Z - 26 \text{ MeV}\,, \quad \Gamma \approx \Gamma_Z(\mathsf{s} = M_Z^2) - 1.2 \text{ MeV}\,. \tag{4}$$

There are other channels in addition to the Z-channel to which the initial and final state of the LEP experiment can couple, e.g., the photon channel and additional channels of which the phase shifts are assumed non-resonant. This means we have a double multichannel resonance [10] with background

$$e\bar{e} \to \begin{matrix} Z \\ \gamma \end{matrix} \to f\bar{f} + n\gamma\,. \tag{5}$$

The partial wave amplitude is a superposition of the Z-boson Breit-Wigner (3), the "γ-Breit-Wigner" and a slowly varying background amplitude $B(\mathsf{s})$ (constant in the Z energy region):

$$a_j(\mathsf{s}) = \frac{R_Z}{\mathsf{s} - \mathsf{s}_R} + \frac{R_\gamma}{\mathsf{s}} + B(\mathsf{s})\,. \tag{6}$$

With the amplitude (6), the S-matrix approach and the Standard Model (on-shell) approach, (using in place of (3) the expression (1) for the Z-boson propagator in (6)), led to similar formulas for the total cross section and the asymmetries, except for the energy independence of the width Γ in the S-matrix approach [1]. These formulas in both approaches contain the Z-Breit-Wigner, the photon term ("γ-Breit-Wigner") and the $Z - \gamma$ interference term which is important for the fits of various asymmetries. Fits of these formulas for the two different approaches to the experimental cross sections and asymmetries were equally good. They led to equally accurate fitted values for mass and width in both approaches, which differed by the expected mass shift (4) [1,11–13]. The experimental data for the Z-boson can not discriminate between the two different definitions of the Z-mass and width.

Though the phenomenological ansatz can be justified in both approaches, theoretically, the on-shell definition of the Standard Model [14] and the pole definition of the S-matrix theory [15] are worlds apart. In the latter case, the resonance is an elementary particle characterized (in addition to its spin j (and internal or channel or resonance species quantum numbers)) by the complex number s_R, and differs from the corresponding definition of a stable particle (bound state pole) just by a non-zero complex part [15]. In the former case, the resonance is a complicated phenomenon which cannot be defined by a number, real or complex. Theoretically, the S-matrix definition has the advantage of gauge invariance and there does not seem to be a consensus whether the *on-shell* definition of M_Z can be gauge invariant. But, besides the on-shell renormalization scheme, there are other renormalization schemes, including the one based on the complex valued position of the propagator pole, and many more different ones which lead to gauge invariant (M_Z', Γ_Z')'s [16].

The definition of resonance mass and width in (perturbation theory of) the Standard Model remains ambiguous unless some further stipulations are added. Therefore, after the above reviewed developments, the popular opinion appears to have changed in favor of the S-matrix definition of M and Γ. However, even for the S-matrix definition by the complex number $\mathsf{s}_R = \left(M_R - i\frac{\Gamma_R}{2}\right)^2$, the mass and width of the Z resonance are not uniquely defined [2]. Conventionally and equivalently one often writes

$$\mathsf{s}_R \equiv \bar{M}_Z^2 - i\bar{M}_Z\bar{\Gamma}_Z = M_R^2\left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right) - iM_R\Gamma_R \tag{7}$$

and calls $\bar{M}_Z = M_R\sqrt{1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2}$ the resonance mass and $\bar{\Gamma}_Z = \Gamma_R\left(1 - \frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right)^{-1/2}$ its width [3].

The insight acquired from the investigation of the line shape problems of the Z-boson influenced the ideas about hadron resonances [17]. The conventional approach [3] for hadron resonances has also been to parameterize the amplitude in terms of a Breit-Wigner (1) with energy dependent width $\Gamma_h(\mathsf{s})$ (which is not as simple as (2) but depends upon the model used for the energy dependence and the definition of M_h). However there has been an ongoing "pole-emic" in favor of the S-matrix pole definition of hadron resonances [18] and the recent editions of [3] list for the baryon resonances like the Δ_{33} the values of the conventional parameters $M_h (= 1232\,\mathrm{MeV for}\Delta)$ and $\Gamma(M_h)(= 120\,\mathrm{MeV for}\Delta)$ as well as the pole position $\mathsf{s}_h\left(= \left(1210 - i\frac{100}{2}\right)\text{ MeV}\right)^1$. When both approaches, the conventional one based on (1) and the S-matrix approach based on (3), were applied to the ρ-meson data [17,19] and compared with each other, the conclusion was that the S-matrix definition of m_ρ and Γ_ρ is phenomenologically preferred. The reason given was that these fitted parameters remained largely independent of the parameterization of the background term $B(\mathsf{s})$ and the $\rho - \omega$ interference. A similar fit to the S-matrix Breit-Wigner (3) was performed for the experimental data on πp scattering in the Δ resonance region [20]. Again the fitted values for the pole definition (3) of M_R and Γ are independent of the background parameterization and significantly smaller than the conventional values from (1). The interpretation of [18] is that the pole position s_h belongs to the Δ-resonance whereas the conventional parameters $\left(M_h,\, \Gamma(M_h^2)\right)$ belong to the Δ together with a large background.

We will give in this paper a definition which completely fixes the ambiguity of the mass and width definition of a relativistic resonance or quasistationary elementary particle. This definition is based on the requirement that the width Γ in the Breit-Wigner energy dis-

[1] Though they still call the Breit-Wigner with energy dependent width (1) the "better form" than the Breit-Wigner (3) given by the pole

tribution should always be exactly equal to the inverse lifetime τ of the exponential decay law, i.e., $\Gamma = \hbar/\tau$. In ordinary quantum mechanics (Hilbert space theory), τ cannot even be defined properly, because Hilbert space mathematics does not allow the exponential law for any state evolving by a self-adjoint Hamiltonian H [21] with a semi-bounded spectrum. Fermi [22] extended the integration over the energy (frequency in his case) from the lower bound ($E = E_0 \equiv m_e + m_{\bar{e}}$ in the present case) to $E = -\infty$. With this assumption for the energy range, these Hilbert space problems are overcome and the Breit-Wigner $(E - (E_R - i\Gamma/2))^{-1}$ as well as (3) above can be related to the exponential $e^{-iE_Rt}e^{-\Gamma_R t}$ by a Fourier transformation (but for $t > 0$ only). This is done in many elementary textbooks (see e.g., (5.118) of [23]). Though numerically the difference between (3) for $(m_e + m_{\bar{e}})^2 \leq \mathsf{s} < +\infty$ and for $-\infty < \mathsf{s} < +\infty$ is small for small values of Γ/M_R ($\approx 10^{-2} \cdots 10^{-15}$) just extending E (or s) to $-\infty$ will violate the stability of matter condition which requires that the Hilbert space be $L^2(\mathbb{R}_{E-E_0>0})$. However, the pole at s_R is in the second Riemann sheet of the S-matrix, and if we take for s of (3) the values $-\infty_{II} < \mathsf{s} < +\infty$ in the second sheet we have avoided the conflict between Fermi's assumption and the semi-boundedness of the energy spectrum. This, however, means that one has to go beyond the Hilbert space $L^2(\mathbb{R}_{E-E_0>0})$. The vector with the energy distribution of (3), the Gamow ket ψ^G (see (18) below), is a functional like the Dirac ket of the Lippmann-Schwinger equation $|E^-\rangle$ and requires the Rigged Hilbert Space. The "ideal" (that means extended to $\mathsf{s} \to -\infty_{II}$) Breit-Wigner in (3) and the "ideal" exponential $e^{-\Gamma t}$ (that means t restricted to $t > 0$) are exact manifestations of the resonance or quasistable particle state, and the Γ of the exact exponential law $e^{-\Gamma t/\hbar} = e^{-t/\tau}$ is now precisely the same as the Γ_R in the exact Breit-Wigner (3). This is a different idealization from von Neumann's idealization in the (complete) Hilbert space where the time dependence of the decay rate can be approximately exponential for "intermediate" times [21] only and where the Breit-Wigner energy distribution can only be an approximation[2]. The widely accepted width-lifetime relation can in ordinary quantum mechanics only be an approximate relation $\Gamma \approx \hbar/\tau$ between approximately defined quantities Γ and τ and has only been justified [24] as a (Weisskopf-Wigner [25]) approximation.

The Rigged Hilbert Space idealization fixes Γ precisely as Γ_R of (3) and (7) because only $\Gamma_R = -2\,\mathrm{Im}\sqrt{\mathsf{s}_R}$ (and not $\bar{\Gamma}_Z$ of (7) or Γ_Z of (1) or any other Γ'_Z) fulfills $\Gamma_R = \hbar/\tau$ and then it fixes the definition of the resonance mass as $M_R = \mathrm{Re}\,\sqrt{\mathsf{s}_R}$. With the Breit-Wigner (3) as the ideal line shape of a relativistic resonance the location of the pole s_R could in principle be extracted precisely from the experimental data.

The problem in all these experimental analyses is to isolate the resonance from the background $B(\mathsf{s})$ and from other resonance terms of (6). This is a practical problem due to the initial and final state photonic corrections and the apparatus resolution, but it is also a problem of principle because even the unfolded "basic cross sections σ^0" may contain interference with some background. One can make the argument that in principle an unstable microphysical state cannot be isolated by a macroscopic apparatus. The prepared in-state ϕ^+ is a superposition (at ideal) of a resonance state ψ^{G} and a background ϕ^{bg}: $\phi^+ = \psi^{\mathrm{G}} + \phi^{\mathrm{bg}}$ [26]. The resonance state ψ^{G} is elementary and characterized, in addition to the spin j_R, by a complex square mass, $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$, $\psi^{\mathrm{G}} = \psi^{\mathrm{G}}_{j_R\mathsf{s}_R}$, in the same way as the stable state is characterized by spin j and real mass-squared m^2, ψ_{jm}, and the vector ϕ^{bg} represents the non-resonant part and is something complicated that changes with ϕ^+ from experiment to experiment. In the scattering amplitude it is represented by $B(\mathsf{s})$. This introduces an ambiguity in the analysis of the experimental data that allows for other theoretical definitions of mass and width. But from this one should not conclude that mass and width of a resonance are defined as technical parameters only which could change with the renormalization scheme. Spin and mass have a fundamental meaning for stable relativistic particles and there is no reason that spin, mass and lifetime should not also have a fundamental meaning for quasistable relativistic particles, even though it is only defined by an idealization, as long as it is the "right" idealization.

For stable elementary particles we have a vector space description defined by the irreducible representation spaces of the Poincaré group $\mathcal{P}$ [27] (from which one then can construct fields [28]). This definition has so far no counterpart for the unstable relativistic particles.

In order to consider an unstable particle such as the Z-boson as a fundamental elementary particle in the Wigner sense, we want to consider in this paper a class of representations of the Poincaré group characterized by a complex eigenvalue $M_R - i\Gamma/2$ of the invariant mass operator $M = (P_\mu P^\mu)^{1/2}$, where M_R is the mass of the unstable particle and Γ, its width. The state vectors of the unstable particle are by definition elements of a representation space of the Poincaré group $\mathcal{P}$. These representations of $\mathcal{P}$ are "minimally complex" in which the Lorentz subgroup is unitary. They are characterized by the numbers (j, s_R) where j is an integer or half integer and $\mathsf{s}_R = (M_R - i\Gamma_R/2)^2$ is a complex number with $M_R > 0$ and $\Gamma_R > 0$[3]. The limit case $\Gamma = 0$ are the unitary irreducible representations of Wigner (j, M_R) describing the stable elementary particle with spin j and mass M_R.

This definition by the representation $\left(j, M_R - i\frac{\Gamma_R}{2}\right)$ of the space-time symmetry group $\mathcal{P}$ is intimately connected with the second definition by the pole of the j-th partial S-matrix element at $\mathsf{s} = \mathsf{s}_R$. In fact we will define $\psi^{\mathrm{G}}_{j\mathsf{s}_R}$'s as the eigenkets of the self-adjoint, invariant square mass operator $P_\mu P^\mu$ with generalized complex eigenvalue s_R which are connected with the S-matrix pole at $\mathsf{s} = \mathsf{s}_R$. We will call these vectors relativistic Gamow kets.

[2] The exact Breit-Wigner cannot be in the domain of the Hamiltonian

[3] There are corresponding representations for $\mathsf{s}_R = (M_R + i\Gamma_R/2)^2$ M_R, $\Gamma_R > 0$

This definition will therefore have features that are the same as those of the pole definition. In particular, the invariant energy wave function (as a function of s) for the resonance state $\psi^{\mathrm{G}}_{j\mathsf{s}_R}$ will be the Breit-Wigner amplitude (3) (i.e., $\langle^{-}\mathsf{s}j|\psi^{\mathrm{G}}_{j\mathsf{s}_R}\rangle \sim a_j(\mathsf{s})$ of (3)). This means the s-distribution $\left|\langle^{-}\mathsf{s}j|\psi^{\mathrm{G}}_{j\mathsf{s}_R}\rangle\right|^2$ of the resonance state vector $\psi^{\mathrm{G}}_{j\mathsf{s}_R}$ is a Breit-Wigner with maximum at $\mathsf{s} = \bar{M}_Z^2 = M_R^2\left(1-\frac{1}{4}\left(\frac{\Gamma_R}{M_R}\right)^2\right)$ and full width at half maximum $2M_R\Gamma_R = 2\bar{M}_Z\bar{\Gamma}_Z$. Usually one calls $\bar{M}_Z$ the mass of the relativistic resonance and $\bar{\Gamma}_Z$ its width [3]. Since the experiment always prepares $\phi^{+} = \psi^{\mathrm{G}} + \phi^{\mathrm{bg}}$, i.e., resonance state with a background, the s-distribution of the (corrected) cross-sections σ_j^0 are given by the modulus of something like (6) with an undetermined background $B(\mathsf{s})$. This makes it difficult to determine the parameters $M_R\,\Gamma_R$ accurately. In addition the complex pole position s_R by itself does not define mass and width separately. Therefore a more specific definition is needed that distinguishes between the different M's and Γ's. This is the definition by the Gamow vector $\psi^{\mathrm{G}}_{j\mathsf{s}_R}$, that has features in terms of which another definition of the quantity Γ can be given. These features are the decay probability $\mathcal{P}(t)$, the total decay rate $\dot{\mathcal{P}}(t)$, and the partial decay rates $\dot{\mathcal{P}}_\eta(t)$, and their exponential laws which defines the lifetime τ. The time dependence of $\mathcal{P}(t)$, $\dot{\mathcal{P}}(t)$ and $\dot{\mathcal{P}}_\eta(t)$ follow from the time evolution of the decaying state $\psi^{\mathrm{G}}_{j,M_R-i\Gamma/2}$ [29], whose time evolution, if exponential, could therefore provide another definition of Γ by demanding that $\Gamma \equiv \frac{\hbar}{\tau}$.

These features were not discussed in connection with the Z-boson and hadron resonances, because for their values of Γ/M they are not observable. The decay rate and the partial decay rates as functions of time are the main focus of experimental investigations for other unstable particles with $\Gamma/M_R \approx 10^{-14}$, like the K^0 [30]. Though in the phenomenological treatment [30,31] of decaying state vectors one is not much concerned with questions of the relativistic definition or the exponential decay law or the line width, it would be still very satisfying if there is a precise vector space description based on the representation (j,s_R) of the relativistic space-time symmetry group $\mathcal{P}$ which is compatible with the S-matrix pole definition of a relativistic resonance, and has all the desired features of a relativistic quasistable particle. The definition of a relativistic resonance or unstable particle by $\psi^{\mathrm{G}}_{j\mathsf{s}_R}$ gives the meaning of a fundamental relativistic particle to the Z-boson, which can be considered as isolated from its background ϕ^{bg}. To what extent such an idealized ket-state can be experimentally prepared is a different question. The accuracy with which the exponential law has been observed in some cases [32] shows that the isolation of the microphysical state ψ^{G} from a background ϕ^{bg} can be very good.

In the remainder of this paper we discuss the mathematical formulation of the rigged Hilbert space quantum mechanics and the state vector representation of resonances and quasi-stable particles. The above phenomenological analysis of the Z-boson and other resonaces is further developed in [33].

2 From the non-relativistic to the relativistic Gamow ket

Gamow kets $\psi^{\mathrm{G}} = |z_R^-\rangle\sqrt{2\pi\Gamma}$, $z_R = E_R - i\Gamma/2$, were introduced in non-relativistic quantum mechanics two decades ago [34] in order to derive a Golden Rule for the time dependent decay rates $\dot{\mathcal{P}}_\eta(t)$ which at $t=0$ goes into Dirac's Golden rule if one makes the following (Born) approximation

$$\langle E|V|\psi^G\rangle \approx \langle E|V|f^D\rangle \qquad E_R \approx E_D\,, \quad \frac{\Gamma}{2E_R} \approx 0\,. \quad (8)$$

Here ψ^G is the eigenket of the Hamiltonian with interaction $H = H_0 + V$ and f^D is the eigenvector of the unperturbed Hamiltonian H_0

$$H\psi^G = (E_R - i\Gamma/2)\psi^G \qquad H_0 f^D = E_D f^D\,. \quad (9)$$

The Gamow kets are like Dirac-Lippmann-Schwinger kets $|E^-\rangle$, functionals of a Rigged Hilbert Space:

$$\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times : \;\; \psi^G = |z_R^-\rangle\sqrt{2\pi\Gamma} \in \Phi_+^\times, \quad |E^-\rangle \in \Phi_+^\times. \quad (10)$$

The generalized eigenvectors, $|E^\pm\rangle = |E, b^\pm\rangle = |E, jj_3^\pm\rangle$, $|z_R^-\rangle$ etc., of the self-adjoint (semi-bounded) energy operator H are mathematically defined by

$$\begin{aligned}\langle H\psi|E^-\rangle &\equiv \langle\psi|H^\times|E^-\rangle \\ &= E\langle\psi|E^-\rangle \;\text{ for all }\; \psi\in\Phi_+, \qquad (11\text{a})\\ \langle H\psi|z_R^-\rangle &\equiv \langle\psi|H^\times|z_R^-\rangle \\ &= z_R\langle\psi|z_R^-\rangle \;\text{ for all }\; \psi\in\Phi_+. \qquad (11\text{b})\end{aligned}$$

The labels b, which could be the angular momentum $j\,,j_3$, are the degeneracy quantum numbers which we shall omit whenever possible. The difference between (11a) and (11b) is that E for the Dirac-kets is the real scattering energy and z_R for the Gamow kets is the complex pole position. The conjugate operator $H^\times$ of the Hamiltonian H is *uniquely* defined by the first equality in (11) as the extension of the Hilbert space adjoint operator $H^\dagger$ to the space of functionals $\Phi_+^\times$ [4] (i.e., on the space $\mathcal{H}$, the operators $H^\times$ and $H^\dagger$ are the same). We shall write (11) also in the Dirac way as

$$H^\times|E^-\rangle = E|E^-\rangle\,; \quad H^\times|z_R^-\rangle = (E_R - i\Gamma/2)|z_R^-\rangle\,. \quad (12)$$

The Dirac kets $|E\rangle$ in (8) are eigenkets of the unperturbed Hamiltonian, $H_0|E\rangle = E|E\rangle$, and E_D is a discrete point embedded in the continuous spectrum $0 < E < \infty$ of H_0.

[4] For (essentially) self-adjoint H, $H^\dagger$ is equal to (the closure of) H; but we shall use the definition (11b) also for unitary operators $\mathcal{U}$ where $\mathcal{U}^\times$ is the extension of $\mathcal{U}^\dagger$, and not of $\mathcal{U}$

In the quantum theory of scattering and decay, the pair of so called in- and out- "states" $|E^+\rangle$ and $|E^-\rangle$, which are solutions of the Lippmann-Schwinger equation,

$$|E^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i0} V|E\rangle = \Omega^\pm|E\rangle\,, \tag{13}$$

are well accepted quantities, though their mathematical properties do not fit into the standard Hilbert space theory. The modulus of the energy-wave function of the prepared in-state ϕ^+, $|\langle^+E|\phi^+\rangle|^2 = |\langle E|\phi^{in}\rangle|^2$, gives the energy distribution in the incident beam of a scattering experiment, and the energy resolution of the observed out-state ψ^-, $|\langle^-E|\psi^-\rangle|^2 = |\langle E|\psi^{out}\rangle|^2$, describes (for perfect efficiency) the energy resolution of the detector.

The sets $\{|E^\pm\rangle\}$ are the basis systems that is used for the Dirac basis vector expansion of the in-states $\phi^+ \in \Phi_-$ and the out-states (observables) $\psi^- \in \Phi_+$ of a scattering experiment

$$\psi^- = \sum_b \int_0^\infty dE|E, b^-\rangle\langle^-E, b|\psi^-\rangle$$
$$\phi^+ = \sum_b \int_0^\infty dE|E, b^+\rangle\langle^+E, b|\phi^+\rangle\,. \tag{14}$$

where b are the degeneracy labels. If one also includes the center-of-mass motion in the description of the states, then b will also include the center-of-mass momentum. The Dirac-Lippmann-Schwinger kets $|E^\pm\rangle$ are in our Rigged Hilbert Space quantum theory antilinear functionals on the spaces $\Phi_\mp$, i.e., they are elements of the dual spaces: $|E^\pm\rangle \in \Phi^\times_\mp$ (see e.g., Sec. III of [35]).

This leads to two Rigged Hilbert Spaces for one and the same Hilbert space $\mathcal{H}$. The two Rigged Hilbert Spaces allow us to formulate the following *new hypothesis* for our quantum theory which will turn out to include asymmetric time evolution:

The pure out-states $\{\psi^-\}$ of scattering theory, which are actually observables as defined by the registration apparatus (detector) are vectors

$$\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times\,. \tag{15a}$$

The pure in-states $\{\phi^+\}$ which are prepared states as defined by the preparation apparatus (accelerator) are vectors

$$\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times\,. \tag{15b}$$

This new hypothesis–with the appropriate choice for the spaces Φ_+ and Φ_- given below in (17)–is essentially all by which our quantum theory differs from the standard Hilbert space quantum mechanics, which imposes the condition $\{\psi^-\} = \{\phi^+\} = \mathcal{H}$ (or $\{\psi^-\} = \{\phi^+\} \subset \mathcal{H}$). As a consequence of this Hilbert space condition, the time evolution generated by the self-adjoint Hamiltonian $\bar{H}$ is a unitary (and therefore reversible) group evolution $U(t) = e^{iHt}$ $-\infty < t < +\infty$.

The time evolution in the spaces Φ_+ of (15a) generated by the essentially self-adjoint Hamiltonian H_+ (which is the restriction of the self-adjoint (closed) $\bar{H}$ to the dense subspace Φ_+) is not a unitary group, but only a semigroup $U_+(t) = e^{iH_+t}$, $0 \le t < \infty$. The time evolution in $\Phi_+^\times$ given by $(U_+(t))^\times = e^{-iH_+^\times t}$ (where the conjugate $U^\times$ is defined as in (11)) is consequently also only a semigroup $0 \le t < \infty$. Similar statements hold for (15b) with $-\infty < t \le 0$[5]. This asymmetric time evolution is a consequence of the time asymmetric boundary condition (15) and not a time asymmetry of the dynamical equation, which is still the Schroedinger or von Neumann differential equation. This time asymmetry has always been tacitly contained in the Lippmann-Schwinger integral equations without however specifying the spaces $\Phi_\pm^\times$ of the solutions $|E^\mp\rangle$ and without giving them an unequivocal physical interpretation as in (15).

This quantum mechanical time asymmetry has been discussed elsewhere [35] and has been mentioned here only to elucidate the time evolution of the Gamow vectors mentioned below. The semigroup $\{U_+(t)\}$ is a restriction to Φ_+ of the unitary group $\{U(t)\}$ in $\mathcal{H}$ and the semigroup $\{U_+^\times(t)\}$ is an extension of the same unitary group $\{U^\dagger(t)\}$ to $\Phi_+^\times$. It is important to record that the unitary group $U(t)$ in $\mathcal{H}$ is not an extension in the sense of Sz.-Nagy of the semigroup $U_+(t)$ on Φ_+ [36,37]. Φ_+ is a complete topological space but not a Hilbert space, and $\mathcal{H}$ is not an extension of Φ_+ as in Sz.-Nagy theory; rather, $\mathcal{H}$ results as the completion of Φ_+ with respect to the scalar product norm[5].

To obtain the non-relativistic Gamow kets one analytically continues the Dirac-Lippmann-Schwinger ket $|E, j, j_3^\pm\rangle$ into the second sheet of the j-th partial S-matrix to the position of the resonance pole z_R. As in ordinary scattering theory, one starts with the following S-matrix elements (suppressing the degeneracy quantum numbers $j\, j_3$):

$$\begin{aligned}
&(\psi^{out}, \phi^{out})\\
&= (\psi^{out}, S\phi^{in}) = (\psi^-, \phi^+)\\
&= \int_0^{+\infty}\int_0^{+\infty} dEdE'\langle\psi^{out}|E\rangle\langle E|S|E'\rangle\langle E'|\phi^{in}\rangle\\
&= \int_0^{+\infty} dE\langle^-\psi|E^-\rangle S(E+i0)\langle^+E|\phi^+\rangle\,.
\end{aligned} \tag{16}$$

In order to arrive at the pole position z_R of $S(E)$, we deform the contour of integration through the cut into the lower half of the second sheet of the energy plane. This is not possible for arbitrary elements ψ^- and ϕ^+ of the Hilbert space, and so one has to assume certain analyticity properties of the energy wave-functions $\langle^-E|\psi^-\rangle$ and $\langle^+E|\phi^+\rangle$ that represent ("realize") the vectors ψ^-, ϕ^+. At this point the new Rigged Hilbert Space hypothesis (15) comes into play: The vectors

$\phi^+ \in \Phi_-$ with the physical interpretation of the in-state prepared by the accelerator, and

[5] It is important not to visualize the inclusions of $\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$ like the inclusion of the two-dimensional plane $\mathbb{R}_2$ in the three-dimensional space $\mathbb{R}_3 = \mathbb{R}_2 \oplus \mathbb{R}_1$, because it is more like the inclusion of the rational numbers in the real numbers

$\psi^- \in \Phi_+$ with the physical interpretation of the observable (decay products) registered by the detector,

are mathematically defined by the property of their energy wave functions $\langle^- E|\psi^-\rangle$ and $\langle^+ E|\phi^+\rangle$ of (14). Respectively:

$$\psi^- \in \Phi_+ \text{if and only if} \langle^- E|\psi^-\rangle \in \mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}\,, \quad (17a)$$
$$\phi^+ \in \Phi_- \text{if and only if} \langle^+ E|\phi^+\rangle \in \mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}\,. \quad (17b)$$

where $\mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}^+}$ are well-behaved Hardy class functions [38] in the upper half plane and $\mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}^+}$ are well-behaved Hardy class functions in the lower half plane. The notation $|_{\mathbb{R}_+}$ means the restriction to the positive real line, i.e., the physical values of energy, and $\mathcal{S}$ denotes the Schwartz space. In contrast $\mathcal{H}$ is realized as the space of Lebesgue square integrable functions $L^2[0,\infty) = L^2(\mathbb{R}^+)$. Thus the new hypothesis (15) means that the energy wave functions are not simply Lebesgue square integrable functions[6] as in ordinary quantum mechanics but are much nicer functions that can be analytically continued into the complex plane (lower half second sheet for $\langle^+ E|\phi^+\rangle$ and $\langle\psi^-|E^-\rangle$ and upper half second sheet for $\langle^- E|\psi^-\rangle$ and $\langle\phi^+|E^+\rangle$) and vanish on the infinite semicircle sufficiently fast. The precise mathematical definition [40] is not important here and it suffices to say that the functions of (17) have all the properties needed to deform the contour of integration (16) into the lower half plane second sheet and to obtain, from the integral around the S-matrix pole z_R, the following representation of the Gamow vector:

$$|z_R = E_R - i\Gamma/2, jj_3\,^-\rangle = \frac{i}{2\pi}\int_{-\infty_{II}}^{+\infty} dE|E, jj_3\,^-\rangle \frac{1}{E - z_R}. \quad (18)$$

This equation is understood as a fu $\Phi_+^\times$. This means that it is a relation between the function $\langle\psi^-|E, jj_3^-\rangle$ of E and its value $\langle\psi^-|z_R, jj_3^-\rangle$ at the complex position z_R[7] for all $\psi^- \in \Phi_+$ (i.e., for observables ψ^- only and not for in-states $\phi^+ \in \Phi_-$). The integral is taken over all values of E along the real axis in the second sheet right below the cut from $E_0(=0)$ to ∞, of which the values $-\infty < E_{II} < 0$ are unphysical, but for which $\langle\psi^-|E^-_{II}\rangle = \langle\psi^-|E^-\rangle$ for the physical values of E along the upper edge of the cut in the first sheet, $0 \le E < \infty$. As a consequence of the Hardy class property, $\langle\psi^-|z^-\rangle$ for any z in the lower half plane is already determined by its values $\langle\psi^-|E^-\rangle$ on the positive semi-axis, i.e., at physical values $0 \le E < \infty$ for which $|\langle\psi^-|E^-\rangle|^2$ is the detector resolution function. The representation (18) is the reason why we have a Breit-Wigner $\frac{1}{E-z_R}$ that extends over $-\infty < E < +\infty$, in spite of the fact that the physical values (i.e., the spectrum of the self-adjoint H) are bounded from below. The same will hold for the relativistic Breit-Wigner in (3).

All the features described here for the non-relativistic case carry over directly to the relativistic case if one replaces the energy (in the center-of-mass frame) E by the relativistic invariant mass square variable (Mandelstam variable) $\mathsf{s} = E^2 - \mathbf{p}^2 = (p_1 + p_2 + \cdots + p_n)^2$ where $p_1\,, p_2 \cdots$ are the momenta of the (two) decay products R. The problem that remains to solve is what to do about the momentum $\mathbf{p}$ which becomes complex when s is taken to complex values.

The Gamow ket $|z_R, jj_3^-\rangle$ as well as the Dirac-Lippmann-Schwinger kets $|E, jj_3^-\rangle$ do not contain the (trivial) center-of-mass motion, this E (and the exact Hamiltonian H) does not include the center-of-mass energy $\frac{\mathbf{p}^2}{2m} = E^{tot} - E$. To obtain the basis system for the space of the center-of-mass plus relative motion in non-relativistic physics one takes the direct product with the eigenket $|\mathbf{p}\rangle$ of the center-of-mass momentum $P = P^1 + P^2$

$$|E\mathbf{p}, jj_3^-\rangle = |\mathbf{p}\rangle \otimes |E, jj_3^-\rangle\,; \quad |z_R\mathbf{p}, jj_3^-\rangle = |\mathbf{p}\rangle \otimes |z_R, jj_3^-\rangle \quad (19)$$

Since in the non-relativistic physics changing of $\mathbf{p}$ (Galilei transformation into a moving frame) does not effect E but only $\frac{\mathbf{p}^2}{2m}$, an analytic extension of E to complex values z does not lead to complex momenta. This is not the case for Lorentz transformations. Complex values of $\mathsf{s} = p_\mu p^\mu$ also means complex values of $E^{tot} = p^0$ and $p^m\,, m = 1,2,3\,;$ because Lorentz transformations intermingle energy and momenta. In order to stay as closely as possible to the non-relativistic case we will consider a special class of "minimally complex" irreducible representations of $\mathcal{P}$. Our construction will lead to complex momenta p^μ, but these momenta will be "minimally complex" in such a way that the 4-velocities $\hat{p}_\mu \equiv \frac{p_\mu}{m}$ remain real. This construction is motivated by a remark of D. Zwanziger [41] and is based on the fact that the 4-velocity eigenvectors $|\hat{\mathbf{p}}j_3(m,j)\rangle$ furnish as valid a basis for the representation space of $\mathcal{P}$ as the usual Wigner basis of momentum eigenvectors $|\mathbf{p}j_3(m,j)\rangle$. When used properly as basis vectors, their introduction does not constitute an approximation. The $|\hat{\mathbf{p}}, j_3\rangle \in \Phi^\times$ are the eigenkets of the 4-velocity operators $\hat{P}_\mu = P_\mu M^{-1}$ and $\phi_{j_3}(\hat{\mathbf{p}}) \equiv \langle j_3\hat{\mathbf{p}}|\phi\rangle$ represents the 4-velocity distribution of a state vector ϕ for a particle with spin j and mass m and therewith contains the same information as the standard momentum distribution $\langle\mathbf{p}|\phi\rangle$. The 4-velocity eigenvectors are often more useful as basis vectors than the momentum eigenvectors [42,43].

3 Relativistic Gamow vectors

Relativistic resonances occur in the scattering of relativistic elementary particles, and relativistic quasistationary states decay into two (or more) relativistic particles, e.g.,

[6] One can show [39] that the two triplets of function spaces

$$\mathcal{S} \cap \mathcal{H}^2_\pm|_{\mathbb{R}_+} \subset L^2(\mathbb{R}_+) \subset \left(\mathcal{S} \cap \mathcal{H}^2_\pm|_{\mathbb{R}_+}\right)^\times$$

which "realize" the two triplets of abstract vector spaces (15), are two Rigged Hilbert Spaces (also called Gelfand triplets) of functions. The two Rigged Hilbert Spaces of the in-states $\{\phi^+\}$ and the out-states $\{\psi^-\}$ are mathematically defined as those Rigged Hilbert Spaces whose realizations are the two Rigged Hilbert Spaces of $\mathcal{S} \cap \mathcal{H}^2_-|_{\mathbb{R}_+}$ and $\mathcal{S} \cap \mathcal{H}^2_+|_{\mathbb{R}_+}$ respectively

[7] This is Titchmarsh theorem for Hardy class functions $\langle\psi^-|E^-\rangle \in \mathcal{H}^2_-$

$e\bar{e} \to R \to f\bar{f}\,(f = e, \mu)$. Relativistic resonances and decaying states are described in the direct product space of two (or more) irreducible representations of the Poincaré group [44,45]

$$\mathcal{H} \equiv \mathcal{H}(m_1, 0) \otimes \mathcal{H}(m_2, 0) = \int_{(m_1+m_2)^2}^{\infty} d\mathsf{s} \sum_{j=0}^{\infty} \oplus \mathcal{H}(\mathsf{s}, j)\,. \tag{20}$$

For simplicity, we have assumed here that there are two decay products, $R \to f_1 + f_2$ with spin zero, described by the irreducible representation spaces $\mathcal{H}^{f_i}(m_i, j_i = 0)$. The direct sum resolution for the more general case involving arbitrary spin j_1 and j_2 is treated in [46]. Since the relativistic Gamow vectors will be defined not as momentum eigenvectors but as 4-velocity eigenvectors in the unitary irreducible representation spaces of the direct product of (20) one needs to use the basis vectors $|\hat{\mathbf{p}}_i j_{i3}(m_i j_i)\rangle$ and $|\hat{\mathbf{p}} j_3(wj)\rangle$ with the normalization

$$\begin{aligned} &\langle \hat{\mathbf{p}}' j_3'(w'j')|\hat{\mathbf{p}} j_3(wj)\rangle \\ &\quad = 2\hat{E}(\hat{\mathbf{p}})\delta(\hat{\mathbf{p}}' - \hat{\mathbf{p}})\delta_{j_3' j_3}\delta_{j'j}\delta(\mathsf{s} - \mathsf{s}') \end{aligned} \tag{21}$$

where

$$\hat{E}(\hat{\mathbf{p}}) = \sqrt{1 + \hat{\mathbf{p}}^2} = \frac{1}{w}\sqrt{w^2 + \mathbf{p}^2} \equiv \frac{1}{w}E(\mathbf{p}, w)\,, \quad w = \sqrt{\mathsf{s}}\,.$$

A relativistic resonance occurs in a particular partial wave characterized by its spin value j. Therefore one cannot use the direct product basis vectors

$$|\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2[m_1 m_2]\rangle \equiv |\hat{\mathbf{p}}_1(m_1 0)\rangle \otimes |\hat{\mathbf{p}}_2(m_2 0)\rangle \tag{22}$$

but the basis in which the total angular momentum or resonance spin j is diagonal. These are the kets $|\hat{\mathbf{p}} j_3(wj)\rangle$ which are also eigenvectors of the 4-velocity operators

$$\hat{P}_\mu = (P^1_\mu + P^2_\mu)M^{-1},\; M^2 = (P^1_\mu + P^2_\mu)(P^{1\mu} + P^{2\mu}) \tag{23}$$

with eigenvalues

$$\hat{p}^\mu = \begin{pmatrix} \hat{E} = \frac{p^0}{w} = \sqrt{1 + \hat{\mathbf{p}}^2} \\ \hat{\mathbf{p}} = \frac{\mathbf{p}}{w} \end{pmatrix} \text{ and } w^2 = \mathsf{s}. \tag{24}$$

In here P^i_μ are the momentum operators in the one particle spaces $\mathcal{H}^{f_i}(m_i, s_i)$ with eigenvalues $p^i_\mu = m_i \hat{p}^i_\mu$. The $|\hat{\mathbf{p}} j_3(wj)\rangle$ are given in terms of the direct product basis vectors (22) by

$$\begin{aligned} &|\hat{\mathbf{p}} j_3(wj)\rangle \\ &\quad = \int \frac{d^3\hat{p}_1}{2\hat{E}_1}\frac{d^3\hat{p}_2}{2\hat{E}_2}|\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2[m_1 m_2]\rangle\langle\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2[m_1 m_2]|\hat{\mathbf{p}} j_3(wj)\rangle \end{aligned} \tag{25}$$

for any

$$(m_1 + m_2)^2 \le w^2 < \infty \quad j = 0, 1, \ldots$$

where the Clebsch-Gordan coefficients $\langle\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2[m_1, m_2]|\hat{\mathbf{p}} j_3(wj)\rangle$ are calculated by the same procedure as given in the classic papers [44,45,47] for the Clebsch-Gordan coefficients $\langle\mathbf{p}_1\mathbf{p}_2[m_1 m_2]|\mathbf{p} j_3(wj)\rangle$ for the Wigner (momentum) basis vectors. This has been done in [46], to yield:

$$\begin{aligned} &\langle\hat{\mathbf{p}}_1\hat{\mathbf{p}}_2[m_1, m_2]|\hat{\mathbf{p}} j_3(wj)\rangle \\ &\quad = 2\hat{E}(\hat{\mathbf{p}})\delta^3(\mathbf{p} - \mathbf{r})\delta(w - \epsilon)Y_{jj_3}(\mathbf{e})\mu_j(w^2, m_1^2, m_2^2) \end{aligned} \tag{26}$$

with $\epsilon^2 = r^2 = (p_1 + p_2)^2,\; r = p_1 + p_2$,

The unit vector $\mathbf{e}$ in (26) is chosen to be in the center-of-mass frame the direction of $\hat{\mathbf{p}}_1^{\rm cm} = -\frac{m_2}{m_1}\hat{\mathbf{p}}_2^{\rm cm}$. The coefficient $\mu_j(w^2, m_1^2, m_2^2)$ fixes the δ-function "normalization" of $|\hat{\mathbf{p}} j_3(wj)\rangle$ and is for the normalization (21) given by

$$\left|\mu_j(w^2, m_1^2, m_2^2)\right|^2 = \frac{2m_1^2 m_2^2 w^2}{\sqrt{\lambda(1, (\frac{m_1}{w})^2, (\frac{m_2}{w})^2)}} \tag{27}$$

where λ is defined by [47]

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2(ab + bc + ac). \tag{28}$$

Since the direct product space (20) describes the states of asymptotically free decay products, the basis vectors (25) are the eigenvectors of the free Hamiltonian $H_0 = P^1_0 + P^2_0$

$$H_0^\times|\hat{\mathbf{p}} j_3(wj)\rangle = E|\hat{\mathbf{p}} j_3(wj)\rangle, \quad E = w\sqrt{1 + \hat{\mathbf{p}}^2}. \tag{29}$$

From these free states, the Dirac-Lippmann-Schwinger scattering states involving interactions can be obtained, in analogy to (13) (cf. also [28] Sec. 3.1) by:

$$|\hat{\mathbf{p}} j_3(wj)^\pm\rangle = \Omega^\pm|\hat{\mathbf{p}} j_3(wj)\rangle \tag{30}$$

where $\Omega^\pm$ are the Møller operators. For the basis vectors at rest, (30) is given by the solution of the Lippmann-Schwinger equation

$$|\mathbf{0} j_3(wj)^\pm\rangle = \left(1 + \frac{1}{w - H \pm i\epsilon}V\right)|\mathbf{0} j_3(wj)\rangle. \tag{31}$$

The interacting states $|\mathbf{0} j_3(wj)^\pm\rangle$ are eigenvectors of the exact Hamiltonian $H = H_0 + V$:

$$H^\times|\mathbf{0} j_3(wj)^\pm\rangle = \sqrt{\mathsf{s}}|\mathbf{0} j_3(wj)^\pm\rangle,\; (m_1 + m_2)^2 \le \mathsf{s} < \infty. \tag{32}$$

For arbitrary velocities, the vectors $|\hat{\mathbf{p}} j_3(wj)^\pm\rangle$ are obtained from the basis vectors at rest $|\mathbf{0} j_3(wj)^\pm\rangle$ by the boost (rotation-free Lorentz transformation) $\mathcal{U}(L(\hat{p}))$ whose parameters are the 4-velocities $\hat{p}^\mu$. The generators of the Lorentz transformations are the interaction-incorporating observables

$$P_0 = H,\; P^m,\; J_{\mu\nu}\,. \tag{33}$$

These exact generators of the Poincaré group are related to the free generators of (20) by terms that describe the interactions ([28], Sec. 3.3). For any fixed pair of values $[jw]$, the basis vectors $|\hat{\mathbf{p}} j_3(wj)^\pm\rangle$, or equivalently the $|\mathbf{0} j_3(wj)^\pm\rangle$ when boosted by $U(L(\hat{p}))$, span a unitary irreducible representation space of the Poincaré group with the "exact generators" (33). The relativistic Gamow vector describing the unstable particle derives from these interaction-incorporating Lippmann-Schwinger kets $|\hat{\mathbf{p}}j_3 (wj)^{\pm}\rangle$.

As mentioned above, the unstable particle is that physical entity which gives rise to the simple pole at $\mathsf{s}_R = (M_R - i\frac{\Gamma_R}{2})^2$ on the second sheet of the analytically extended partial wave S-matrix S_{j_R}. Therefore, to obtain the Gamow vectors, and therewith a state vector description of unstable particles, we seek to obtain the analytic extensions of the Dirac-Lippmann-Schwinger kets (30) or (31) to the location of the pole s_R. This requirement imposes the condition that the wave functions of the in-states $\phi^+ \in \Phi_-$ and out-states $\psi^- \in \Phi_+$ have the same analyticity properties in the square mass variable as the energy wave functions of the non-relativistic case synopsized by (17), with the exception that mathematical rigor requires that a closed subspace $\tilde{\mathcal{S}}$ of the Schwartz space, developed in [48], needs to be considered:

$$\begin{aligned} &\psi^- \in \Phi_+ \quad \text{if and only if} \\ &\qquad \langle^-\hat{\mathbf{p}}j_3\mathsf{s}j|\psi^-\rangle \in \left(\tilde{\mathcal{S}} \cap \mathcal{H}^2_+\right)\Big|_{\mathbb{R}_{(m_1+m_2)^2}} \\ &\phi^+ \in \Phi_- \quad \text{if and only if} \\ &\qquad \langle^+\hat{\mathbf{p}}j_3\mathsf{s}j|\phi^+\rangle \in \left(\tilde{\mathcal{S}} \cap \mathcal{H}^2_-\right)\Big|_{\mathbb{R}_{(m_1+m_2)^2}}, \end{aligned} \tag{34}$$

where $\mathbb{R}_{(m_1+m_2)^2} = [(m_1+m_2)^2, \infty)$. The details of this construction of $\tilde{\mathcal{S}}$ will be given in a forthcoming paper. Another requirement for the validity of the analytic continuation is that the s-contour of integration in the completeness relation for (ψ^-, ϕ^+) with respect to the $|\hat{\mathbf{p}}j_3\mathsf{s}j^{\pm}\rangle$ basis, namely

$$\begin{aligned} (\psi^-, \phi^+) = \sum_{jj_3} \int \frac{d^3\hat{\mathbf{p}}}{2\hat{p}^0} \int_{(m_1+m_2)^2}^{\infty} d\mathsf{s} \langle \psi^- | \hat{\mathbf{p}}j_3\mathsf{s}j^-\rangle \\ \times S_j(\mathsf{s}) \langle \hat{\mathbf{p}}j_3\mathsf{s}j^+|\phi^+\rangle \end{aligned} \tag{35}$$

can be deformed into the second sheet of the j_R-th partial S-matrix element $S_j(E)$. With these analyticity requirements, and in complete analogy to the non-relativistic case (18), one deforms the s-contour of integration in (35) so that the amplitude (ψ^-, ϕ^+) separates into a resonance state associated with the pole at s_R and a background term. The pole term yields the kets

$$\begin{aligned} |\hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle &= \frac{i}{2\pi} \int_{-\infty_{II}}^{+\infty} d\mathsf{s} |\hat{\mathbf{p}}j_3(\mathsf{s}j_R)^-\rangle \frac{1}{\mathsf{s} - \mathsf{s}_R}, \\ \mathsf{s}_R &= \left(M_R - i\frac{\Gamma_R}{2}\right)^2 \end{aligned} \tag{36}$$

with the Breit-Wigner s-distribution of (3) that extends from $-\infty_{II} < \mathsf{s} < \infty$. These are the relativistic Gamow kets that we set out to construct.

The relativistic Gamow kets (36) are generalized eigenvectors of the invariant mass squared operator $M^2 = P_\mu P^\mu$ with eigenvalue $\mathsf{s}_R = \left(M_R - i\frac{\Gamma_R}{2}\right)^2$

$$\begin{aligned} &\langle \psi^- | M^2 | \hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle \\ &\quad = \left(M_R - i\frac{\Gamma_R}{2}\right)^2 \langle \psi^- | \hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle \\ &\quad \text{for every} \psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times. \end{aligned} \tag{37}$$

To prove (37) from (36) and also in order to obtain (36) from the pole term of the S-matrix, one needs to use the Hardy class properties (34) of the space Φ_+ [34] and the usual analyticity properties of the S-matrix elements [15]. The continuous linear combinations of the Gamow vectors (36) with an arbitrary 4-velocity distribution function $\phi_{j_3}(\hat{\mathbf{p}}) \in \mathcal{S}$ (Schwartz space),

$$\psi^{\mathrm{G}}_{j_R\mathsf{s}_R} = \sum_{j_3} \int \frac{d^3\hat{p}}{2\hat{p}^0} |\hat{\mathbf{p}}j_3(\mathsf{s}_R, j_R)^-\rangle \phi_{j_R}(\hat{\mathbf{p}}), \tag{38}$$

represent the velocity wave-packets of the unstable particles. As an immediate consequence of the integral resolution (36), they also have a Breit-Wigner distribution $\frac{1}{\mathsf{s}-\mathsf{s}_R}$ in the square mass variable that extends over $-\infty_{II} < \mathsf{s} < +\infty$ as given in (3).

In the vector space spanned by the Gamow kets $|\hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle$, the Lorentz transformations $\mathcal{U}(\Lambda)$ are represented unitarily:

$$\mathcal{U}(\Lambda)|\hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle = \sum_{j_3'} D^{j_R}_{j_3' j_3}(\mathcal{R}(\Lambda, \hat{p})) |\mathbf{\Lambda}\hat{\mathbf{p}}j_3'(\mathsf{s}_R j_R)^-\rangle, \tag{39}$$

where $\mathcal{R}(\Lambda, \hat{p}) = L^{-1}(\Lambda\hat{p})\Lambda L(\hat{p})$ is the Wigner rotation. In particular for the rotation free Lorentz boost $L(\hat{p})$ we have

$$\mathcal{U}(L(\hat{p}))|\hat{\mathbf{p}} = \mathbf{0}, j_3(\mathsf{s}_R j_R)^-\rangle = |\hat{\mathbf{p}}j_3(\mathsf{s}_R j_R)^-\rangle. \tag{40}$$

It is important to remark here that the complexness of the Poincaré invariant $P_\mu P^\mu = \left(\mathsf{s}_R - i\frac{\Gamma_R}{2}\right)^2$ (37), or equivalently that of the momenta $p_\mu = \left(\mathsf{s}_R - i\frac{\Gamma_R}{2}\right)\hat{p}_\mu$, does not upset the unitarity of the $\mathcal{U}(\Lambda)$. The crucial observation is that the parameters of the homogeneous Lorentz transformations (40) are not the momenta p^μ, but the 4-velocities $\hat{p}^\mu = \frac{p^\mu}{w}$, since the boost matrix L is given by

$$L^\mu{}_\nu = \begin{pmatrix} \frac{p^0}{w} & -\frac{p_n}{w} \\ \frac{p^k}{w} & \delta^k_n - \frac{\frac{p^k}{w}\frac{p_n}{w}}{1+\frac{p^0}{w}} \end{pmatrix}, \; L(\hat{p}) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \hat{p}. \tag{41}$$

We choose these parameters $\hat{p}_\mu$ real and they remain real under general Lorentz transformations which are products of boosts and ordinary rotations. The complexness of the momenta is solely due to complexness of the invariant mass $w = \sqrt{\mathsf{s}_R}$.

The analyticity and smoothness properties (34) needed for the construction of the Rigged Hilbert Space theory of non-relativistic Gamow vectors further infer that the time translation of the decaying state is given by a semigroup. For instance, the rest state vectors of the quasistable particle transforms as

$$\begin{aligned} &\mathrm{e}^{-iH^\times t}|\hat{\mathbf{p}} = \mathbf{0}, j_3(\mathsf{s}_R j_R)^-\rangle \\ &\quad = \mathrm{e}^{-im_R t}\mathrm{e}^{-\Gamma_R t/2}|\hat{\mathbf{p}} = \mathbf{0}, j_3(\mathsf{s}_R j_R)^-\rangle \; \text{for} t \geq 0 \text{ only} \end{aligned} \tag{42}$$

where t is time in the rest system. This is the required exponential time evolution which assures the validity of the *exact* exponential law for the partial and total decay rates

$$\dot{\mathcal{P}}(t) = \frac{d}{dt}\mathcal{P}(t) = \frac{\Gamma_R}{\hbar} e^{-\Gamma_R t/\hbar}\,;$$
$$\dot{\mathcal{P}}_\eta(t) = \frac{\Gamma_{R\eta}}{\hbar} e^{-\Gamma_R t/\hbar}\,;\ t \geq 0\,, \qquad (43)$$

where Γ_R is exactly the imaginary part of the generalized eigenvalue of the mass operator M for the Gamow kets in (37) which in turn according to (36) is exactly $-2\mathrm{Im}\sqrt{\mathsf{s}_\mathrm{R}}$ of the pole position s_R in the "ideal" Breit-Wigner (3). The relativistic Gamow vector is the theoretical link that connects the ideal relativistic Breit-Wigner energy distribution of the second sheet S-matrix pole (3) to the exact exponential decay law (43) and justifies the lifetime-width relation $\tau = \frac{\hbar}{\Gamma_R}$ as a precise equality.

4 Conclusion

We have constructed the relativistic Gamow vector in analogy to the non-relativistic Gamow vector which had been defined some time ago in the framework of time asymmetric quantum mechanics in Rigged Hilbert Spaces. Gamow vectors have all the properties needed to represent quasistable states and resonances. They are associated to resonance poles of the S-matrix, have a Breit-Wigner energy distribution which for the relativistic Gamow vector is given by (36) leading to the scattering amplitude (3), and have an exact exponential time evolution (42) guaranteeing the exponential law (43). Then the connection between the width Γ_R measured by (3) and the lifetime $\tau = \frac{\hbar}{\Gamma_R}$ measured by the exponential law (43) holds exactly. This relation $\tau = \frac{\hbar}{\Gamma}$ cannot be obtained from (1) for $\Gamma = \Gamma_Z$ since the definition of the Gamow vectors (36) requires the denominator of (3). It is quite unlikely that a state vector (or state operator) can be associated to (1) since the Hardy class Rigged Hilbert Spaces (34), from which the Gamow vector (36) is derived, have a very special and tight mathematical structure.

If one wants this lifetime-width relation, $\tau = \frac{\hbar}{\Gamma}$, to hold universally and exactly, then Γ must be the Γ_R defined by (3) and not the more commonly used $\bar{\Gamma}_Z$ of (7) nor the standard Γ_Z of (1). The "resonance mass" is then given from the inverse lifetime Γ_R and the S-matrix pole position s_R as $\mathrm{Re}\sqrt{\mathsf{s}_\mathrm{R}} = \mathrm{M_R}$ which differs from the standard $M_Z \approx M_R + 26\,\mathrm{MeV}$ and from $\bar{M}_Z \approx M_R + 8\,\mathrm{MeV}$.

Defining the relativistic resonance and quasistable relativistic particle by the Gamow vector puts the quasistable and stable elementary particles on a more equal footing. Stable elementary particles are defined by irreducible unitary representation (j, m^2) spaces of the Poincaré group $\mathcal{P}$ [27]. The Dirac-Lippmann-Schwinger kets $|\hat{\mathbf{p}} j_3 (\mathsf{s} j)^-\rangle$ in (30) are basis vectors of an irreducible unitary representation (j, s) of $\mathcal{P}$ [28]. The Gamow kets $|\hat{\mathbf{p}} j_3 (\mathsf{s}_R j)^-\rangle$ take this just one small step further because they are obtained from the "out-states" $|\hat{\mathbf{p}} j_3 (\mathsf{s} j)^-\rangle$ by analytic continuation to the S-matrix pole position s_R. The Gamow kets $|\hat{\mathbf{p}} j_3 (\mathsf{s}_R j)^-\rangle$ are also a basis system of a representation (j, s_R) of Poincaré transformations. But these transformations form only the semigroup of the Poincaré transformations into the forward light cone $\mathcal{P}_+$, of which the time translations at rest for $t > 0$, (42), are special examples. The representations $(j, \mathsf{s}_R) = \left(j, M_R - i\frac{\Gamma_R}{2}\right)$ of $\mathcal{P}_+$ are "minimally complex" representations in which the Lorentz subgroup is unitary. They are characterized by the integer or half-integer j and by $M_R > 0$ and $\Gamma_R > 0$. The limit case $\Gamma_R = 0$ are the unitary irreducible representation of Wigner (j, M_R) describing the stable elementary particle with spin j and mass M_R, and thus quasistable and stable particles are just special cases of representations of Poincaré transformations[8].

The relativistic Gamow vectors unify stable and quasistable relativistic particles; the Z-boson now becomes a fundamental particle in the sense of Wigner, like the proton. Stable particles are representations characterized by a real mass and have unitary group time evolutions. Quasistable and resonance particles are semigroup representations characterized by a complex mass and have semigroup time evolutions. This time asymmetry on the microphysical level is the most surprising and remarkable property of relativistic Gamow vectors.

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